Eigenvalues of Cayley graphs

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Cayley graphs

Definition

Let X be a group. Let $S \subset X$ be such that $1 \notin S$, $s^{-1} \in S$ whenever $s \in S$, and $\langle S \rangle = X$.

The Cayley graph $\mathrm{Cay}(X, S)$ is defined to have vertex set X such that $x, y \in X$ are adjacent if and only if $x^{-1}y \in S$.

- Cayley graphs are significant objects of study in algebraic graph theory.
- They are also useful in many applications, e.g. constructions of interconnection networks and expander graphs, etc.
- A connected graph G is isomorphic to a Cayley graph iff $\mathrm{Aut}(G)$ contains a subgroup that is regular on $V(G)$.

eigenvalues of Cayley graphs

- It is interesting to study eigenvalues of Cayley graphs.
- It is crucial to estimate the second largest eigenvalues of Cayley graphs in order to construct expanders.
- It is well known that the spectrum of a Cayley graph can be expressed in terms of characters of the underlying group.

Definition

A representation of a group X is a pair (π, V), where V is a vector space over $\mathbb{C}$ and $\pi : X \to \mathrm{GL}(V)$ is a homomorphism. Call $\dim(V)$ the degree of π.

π is irreducible if it leaves no nontrivial subspace of V invariant.

The character of π is the map $\chi_\pi : X \to \mathbb{C}$, $x \mapsto Tr(\pi(x))$.

Theorem

If X is an Abelian group and χ_x $(x \in X)$ are distinct irreducible characters of X, then the eigenvalues of $\mathrm{Cay}(X, S)$ are

$$\chi(S) := \sum_{s \in S} \chi_x(s), \quad x \in X.$$

unitary Cayley graphs

Definition

A circulant graph is a Cayley graph on a cyclic group.

Definition

Define

$$G_n := \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^*),\ n \geq 2$$

and call it the unitary Cayley graph of $\mathbb{Z}_n$, where $\mathbb{Z}_n^* = \{j : 1 \leq j \leq n-1, \gcd(j,n) = 1\}$ is the multiplicative group of units of $\mathbb{Z}_n$.

This notion was introduced (in its general form) in:

[E. D. Fuchs, Largest induced cycles in circulant graphs, Elec. J. Combin. 12 (2005), R52]

where a lower bound on the length of a longest induced cycle in G_n was obtained.

eigenvalues of G_n

Since the characters of $\mathbb{Z}_n$ are χ_k, where

$$\chi_k(a) = e^{2\pi ika/n},\ 0 \leq k \leq n-1,$$

we have:

Proposition

The eigenvalues of G_n are given by the Ramanujan sums

$$\begin{aligned}\lambda_k &= \sum_{1\leq j<n,\, \gcd(j,n)=1} e^{2\pi ikj/n} \\ &= \mu\left(\frac{n}{(n,k)}\right)\frac{\phi(n)}{\phi\left(\frac{n}{(n,k)}\right)},\ \ 0 \leq k \leq n-1\end{aligned}$$

where ϕ is Euler's totient function and μ is the Möbius function (i.e. $\mu(n) = 0$ if n is not square-free, $\mu(n) = 1$ or -1 if n is square-free with an even or odd number of prime factors).

integrality and combinatorial properties of G_n

Definition

A graph is called integral if all its eigenvalues are integers.

Observation

G_n is integral with each nonzero eigenvalue a divisor of $\phi(n)$.

[W. Klotz and T. Sander, Some properties of unitary Cayley graphs, Elec. J. Combin. 14 (2007), R45]

The authors also studied combinatorial and spectral properties (chromatic number, clique number, independence number, diameter, vertex connectivity, perfectness) of G_n. For example,

— G_n is perfect iff n is even, or n is odd with at most two distinct prime factors;

— G_n has both ± 1 as eigenvalues with multiplicity $\phi(n)$ iff n is even and square-free.

energy of G_n

Definition

The energy of a graph G with order n and eigenvalues $\lambda_1, \ldots, \lambda_n$ is defined as

$$E(G) = \sum_{j=1}^{n} |\lambda_j|.$$

Theorem

The energy of G_n is given by

$$E(G_n) = 2^k \phi(n),$$

where k is the number of distinct prime factors of n.

[A. Ilić, The energy of unitary Cayley graphs, Linear Alg. Appl. 431 (2009) 1881-1889]

Denote by $\overline{G}$ the complement of a graph G.

Theorem

Let $p_1, \ldots, p_k$ be distinct prime factors of n. Then

$$E(\overline{G}_n) = 2n - 2 + (2^k - 2)\phi(n) - \prod_{j=1}^{k} p_j + \prod_{j=1}^{k} (2 - p_j).$$

[A. Ilić, The energy of unitary Cayley graphs, Linear Alg. Appl. 431 (2009) 1881-1889]

Definition

A graph G with n vertices is called hyperenergetic if its energy exceeds the energy of the complete graph K_n, that is, if $E(G) > 2n - 2$.

Theorem

G_n is hyperenergetic iff n has at least three distinct prime factors, or exactly two prime factors both greater than 2.

Theorem

$\overline{G}_n$ is hyperenergetic iff n has at least two distinct prime factors and $n \neq 2p$ for any prime p.

[A. Ilić, The energy of unitary Cayley graphs, Linear Alg. Appl. 431 (2009) 1881-1889]

The formula $E(G_n) = 2^k \phi(n)$ and the result on when G_n is hyperenergetic were also obtained in:

[H. N. Ramaswamy and C. R. Veena, On the energy of unitary Cayley graphs, Elec. J. Combin. 16 (2009), N24]

Definition

A finite d-regular graph G is called Ramanujan if every eigenvalue λ of G other than $\pm d$ satisfies

$$\lambda \leq 2\sqrt{d-1}.$$

Theorem

(Alon-Boppana)
Let (G_i) be a family of finite, connected, d-regular graphs with $|V(G_i)| \to \infty$ as $i \to \infty$. Then

$$\liminf_{i\to\infty} \lambda_2(G_i) \geq 2\sqrt{d-1}.$$

Theorem

G_n is Ramanujan iff n is one of the following (where $p < q$ are odd primes and $s \geq 1$):

- 2^s $(s \geq 2)$;
- p;
- $2^s p$ $(s \geq 1, p > 2^{s-3} + 1)$;
- p^2, $2p^2$ or $4p^2$;
- pq or $2pq$ $(p < q \leq 4p - 5)$;
- $4pq$ $(p < q \leq 2p - 3)$.

[A. Droll, A classication of Ramanujan unitary Cayley graphs, Elec. J. Combin. 17 (2010), N29]

GCD graphs of $\mathbb{Z}_n$

Definition

Given an integer $n \geq 2$ and a set D of positive proper divisors of n, the GCD graph $G_n(D)$ is defined to have vertex set $\mathbb{Z}_n$ such that $a, b \in \mathbb{Z}_n$ are adjacent if and only if $\gcd(a-b, n) \in D$.

Theorem

The eigenvalues of $G_n(D)$ are:

$$\lambda_k(n, D) = \sum_{d \in D} c\left(k, n/d\right),$$

where $c(k,n) = \sum_{1 \leq j < n,\ \gcd(j,n)=1} e^{2\pi ikj/n}$.

Theorem

$G_n(D)$ is integral.

[W. Klotz and T. Sander, Some properties of unitary Cayley graphs, Elec. J. Combin. 14 (2007), R45]

integral circulants = GCD graphs

Theorem

A circulant graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ *is integral iff* S *is a union of some* $S_n(d)$*, where for a proper divisor* d *of* n*,*

$$S_n(d) = \{j : 1 \leq j \leq n-1, \gcd(j,n) = d\}.$$

In other words, a circulant graph on n *vertices is integral iff it is a GCD graph.*

[W. So, Integral circulant graphs, Discrete Math. 306 (2005) 153-158]

energy of $G_n(D)$

Theorem

(a) *If a prime $p|n$ but $p^2 \nmid n$, then*

$$E(G_n(\{1,p\})) = 2^{k-1} p \, \phi\left(\frac{n}{p}\right),$$

where k is the number of prime factors of n.

(b) *If n is square-free with k prime factors, then*

$$E(G_n(\{p,p'\})) = 2^k \phi(n)$$

for distinct prime factors p, p' of n.

[A. Ilić, The energy of unitary Cayley graphs, Linear Alg. Appl. 431 (2009) 1881-1889]

Theorem

(a) *If n is odd, then 4 divides $E(G_n(D))$;*

(b) *if n is even, then 4 does not divide $E(G_n(D))$ if and only if $n/2 \notin D$ and $\sum_{d \in D}(-1)^d \phi(n/d) < 0$.*

[A. Ilić and M. Bašić, New results on the energy of integral circulant graphs, Applied Math. & Comput. 218 (2011) 3470-3482]

In the same paper, the energy of $G_n(D)$ when $|D| = 2$ was also studied, generalizing corresponding results in:

[A. Ilić, The energy of unitary Cayley graphs, Linear Alg. Appl. 431 (2009) 1881-1889]

The energy of GCD graphs with a given prime power order was studied by J. W. Sander and T. Sander in three recent papers.

They determined the minimum energy of such graphs and obtained partial results on the maximum energy of them.

Theorem

Let p^e be a prime power and $D = \{p^{e_1}, \dots, p^{e_k}\}$, where $0 \le e_1 < \cdots < e_k \le e-1$. Then

$$E(G_{p^e}(D)) = 2(p-1)\left(p^{e-1}k - (p-1)\sum_{r=1}^{k-1}\sum_{j=r+1}^{k} p^{e-e_j+e_r-1}\right).$$

[J. W. Sander, T. Sander, The energy of integral circulant graphs with prime power order, Appl. Anal. Discrete Math. 5 (2011) 22-36]

Definition

A set of positive integers is called multiplicative if it is of the form

$$A = \prod_{i=1}^{t} A_i,$$

where $A_i \subset \{1, p_i, p_i^2, \ldots\}$ for distinct primes $p_1, \ldots, p_t$.

[T. A. Le and J. W. Sander, Convolutions of Ramanujan sums and integral circulant graphs, International J. Number Theory 8 (2012) 1777-1788]

Theorem

If D is multiplicative, then

$$E(G_n(D)) = \prod_{p|n} E(G_{p^{e_p(n)}}(D_p)),$$

where $e_p(n)$ is the order of the prime p in n and

$$D_p = \{p^{e_p(d)} : d \in D\}.$$

[T. A. Le and J. W. Sander, Convolutions of Ramanujan sums and integral circulant graphs, International J. Number Theory 8 (2012) 1777-1788]

Theorem

Let $n = p_1^{e_1} \cdots p_k^{e_k} \geq 2$, where $p_1 < \cdots < p_k$.

(a) *The maximum energy of a GCD graph on n vertices with a multiplicative divisor set is equal to $\prod_{j=1}^{k} \theta(p_i^{e_i})$, where $\theta(p^e)$ is a certain function of p and e. Moreover, all multiplicative sets D achieving this maximum energy are determined.*

(b) *The minimum energy of a GCD graph on n vertices with a multiplicative divisor set is equal to $2n\left(1 - \frac{1}{p_1}\right)$. Moreover, all multiplicative divisor sets achieving this minimum energy are characterized.*

[T. A. Le, J. W. Sander, Extremal energies of integral circulant graphs via multiplicativity, Linear Alg. Appl. 437 (2012) 1408-1421]

conjectures on GCD graphs

In the same paper, the authors made the following three conjectures:

Conjecture

The minimum energy of a circulant graph on n vertices is equal to

$$2n\left(1-\frac{1}{p}\right),$$

where p is the smallest prime divisor of n.

Conjecture

If $G_n(D)$ achieves the minimum energy among all GCD graphs of order n, then D must be multiplicative.

The first conjecture is implied by the second one.

Denote

$$D(n) = \{1 \leq d \leq n-1 : d \mid n\}.$$

Conjecture

Let $D_1, D_2 \subseteq D(n)$ be two multiplicative sets. If $G_n(D_1)$ and $G_n(D_2)$ are cospectral, then $D_1 = D_2$.

This is a weakening of the following conjecture due to W. So (2005):

Conjecture

$G_n(D_1)$ and $G_n(D_2)$ are cospectral if and only if $D_1 = D_2$.

The latter conjecture was confirmed when n is a prime power or the product of a prime power and another prime (C. F. Cusanza 2006).

unitary Cayley graphs of finite commutative rings

Definition

Let R be a finite ring with unit element $1 \neq 0$.

Let R^* be the set of units of R.

The unitary Cayley graph is defined to be the Cayley graph $G_R = \mathrm{Cay}(R, R^*)$ on $(R, +)$ with respect to R^*.

That is, G_R has vertex set R such that $x, y \in R$ are adjacent if and only if $x - y \in R^*$.

[E. D. Fuchs, Largest induced cycles in circulant graphs, Elec. J. Combin. 12 (2005), R52]

notation

- A local ring is a commutative ring with a unique maximal ideal M.
- Every finite commutative ring R can be decomposed uniquely (up to reordering of factors) into a direct product of finite local rings, say,

$$R = R_1 \times R_2 \times \cdots \times R_s.$$

- Denote by M_i the unique maximal ideal of R_i and set $m_i = |M_i|$, $1 \leq i \leq s$.
- Assume $|R_1|/m_1 \leq |R_2|/m_2 \leq \cdots \leq |R_s|/m_s$.

structure of G_R

Proposition

(a) $G_R = \otimes_{i=1}^{s} G_{R_i}$ *(tensor product or Kronecker product)*

(b) *Each G_{R_i} is a complete multipartite graph with partite sets the cosets of M_i in R_i.*

(c) *The degree of G_R is equal to*

$$|R^*| = \prod_{i=1}^{s}(|R_i| - m_i) = |R| \prod_{i=1}^{s}\left(1 - \frac{1}{|R_i|/m_i}\right).$$

eigenvalues of G_R

Proposition

(Akbari, et al, 2009; Kiani, et al, 2011)
The eigenvalues of G_R are

(a)

$$\lambda_C = (-1)^{|C|} \frac{|R^*|}{\prod_{j \in C}(|R_j^*|/m_j)}$$

repeated $\prod_{j \in C} |R_j^*|/m_j$ times, where C runs over all subsets of $\{1, 2, \ldots, s\}$; and

(b) 0 *with multiplicity* $|R| - \prod_{i=1}^{s} \left(1 + \frac{|R_i^*|}{m_i}\right)$.

In particular, if R is a finite local ring and m is the order of its unique maximal ideal, then

$$Spec(G_R) = \begin{pmatrix} |R| - m & -m & 0 \\ 1 & \frac{|R|}{m} - 1 & \frac{|R|}{m}(m-1) \end{pmatrix}.$$

energy of unitary Cayley graphs

The following are generalisations of the counterpart results in [A. Ilić 2009].

Theorem

$$E(G_R) = 2^s|R^*|$$

Theorem

$$E(\overline{G}_R) = 2|R| - 2 + (2^s - 2)|R^*| - \prod_{j=1}^{s} |R_j|/m_j + \prod_{j=1}^{s} (2 - |R_j|/m_j)$$

[D. Kiani, et al, Energy of unitary Cayley graphs and gcd-graphs, Linear Alg. Appl. 435 (2011) 1336-1343]

hyperenergetic unitary Cayley graphs

Theorem

G_R is not hyperenergetic iff one of the following holds:

(a) $s = 2$, $|R_1|/m_1 \geq 3$ *and* $|R_2|/m_2 \geq 4$;

(b) $s \geq 3$, *and either* $|R_{s-2}|/m_{s-2} \geq 3$, *or* $|R_{s-1}|/m_{s-1} \geq 3$ *and* $|R_s|/m_s \geq 4$.

[D. Kiani, et al, Energy of unitary Cayley graphs and gcd-graphs, Linear Alg. Appl. 435 (2011) 1336-1343]

The energy of $L(G_R)$ and the spectral moments of G_R and $L(G_R)$ are computed in [X. Liu and S. Zhou 2012].

Ramanujan unitary Cayley graphs

Theorem

G_R is Ramanujan if and only if one of the following holds:

(a) $R_i/M_i \cong \mathbb{F}_2$ $(1 \le i \le s)$;

(b) $R_i \cong \mathbb{F}_2$ $(1 \le i \le s-3)$, $R_i \cong \mathbb{F}_3$ $(s-2 \le i \le s)$;

(c) $R_i \cong \mathbb{F}_2$ $(1 \le i \le s-3)$, $R_{s-2} \cong R_{s-1} \cong \mathbb{F}_3$, $R_s \cong \mathbb{F}_4$;

(d) $R_i \cong \mathbb{F}_2$ $(1 \le i \le s-3)$, $R_{s-2} \cong R_{s-1} \cong R_s \cong \mathbb{F}_4$;

(e) $R_i \cong \mathbb{F}_2$ $(1 \le i \le s-2)$, $R_{s-1} \cong \mathbb{F}_3$, $R_s \cong \mathbb{Z}_9$ *or* $\mathbb{Z}_3[X]/(X^3)$;

[X. Liu and S. Zhou, Spectral properties of unitary Cayley graphs of finite commutative rings. Electronic J. Combin. 19 (2012), P13]

Theorem

(cont'd)

(f) $R_1 \cong \mathbb{Z}_4$ *or* $\mathbb{Z}_2[X]/(X^2)$, $R_i \cong \mathbb{F}_2$ $(2 \leq i \leq s-2)$, $R_{s-1} \cong \mathbb{F}_{q_1}$ *and* $R_s \cong \mathbb{F}_{q_2}$ *for some prime powers* $q_1, q_2 \geq 3$ *with* $q_1 \leq q_2 \leq q_1 + \sqrt{(q_1-2)q_1}$;

(g) $R_i \cong \mathbb{F}_2$ $(1 \leq i \leq s-2)$, $R_{s-1} \cong \mathbb{F}_{q_1}$ *and* $R_s \cong \mathbb{F}_{q_2}$ *for some prime powers* $q_1, q_2 \geq 3$ *with* $q_1 \leq q_2 \leq 2\left(q_1 + \sqrt{(q_1-2)q_1}\right) - 1$;

(h) $R_i/M_i \cong \mathbb{F}_2$ $(1 \leq i \leq s-1)$, $R_s/M_s \cong \mathbb{F}_q$ *for a prime power* $q \geq 3$, *and* $\prod_{i=1}^{s} m_i \leq 2\left(q - 1 + \sqrt{(q-2)q}\right)$.

Ramanujan complements of unitary Cayley graphs

Theorem

(X. Liu and S. Zhou 2012)
$\overline{G}_R$ *is Ramanujan if and only if one of the following holds:*

(a) $|R_i|/m_i = 2$ $(1 \leq i \leq s)$ *and*

$$\prod_{i=1}^{s} m_i \leq 2^{s+1} - 3 + 2\sqrt{2^s(2^s - 3)};$$

(b) $2 = |R_1|/m_1 = \cdots = |R_t|/m_t < |R_{t+1}|/m_{t+1}$ *for some* t $(2 \leq t < s)$ *and* $|R^*| \leq 2\sqrt{|R|} - 3$;

(c) $2 = |R_1|/m_1 < |R_2|/m_2$ *and* $|R^*| \leq 2\sqrt{|R| - 2} - 1$;

(d) $3 \leq |R_1|/m_1$ *and* $\frac{|R^*|}{(|R_1|/m_1)-1} \leq$
$$-(2(|R_1|/m_1) - 3) + \sqrt{(2(|R_1|/m_1) - 3)^2 + (4|R| - 9)}.$$

GCD graphs of UFDs

The following is a generalization of the concept of GCD graphs of $\mathbb{Z}_n$.

Definition

Let R be a unique factorization domain (UFD).

Let $0 \neq c \in R$ be a non-unit element and D a set of proper divisors of c.

The GCD graph $G_{R/(c)}(D)$ is defined to have vertex set $R/(c)$ such that $x + (c), y + (c) \in R/(c)$ are adjacent if and only if $\gcd(x - y, c) \in D$. (The gcd here is unique up to associates.)

In particular, define $G_{R/(c)} := G_{R/(c)}(\{1\})$.

[D. Kiani, et al, Energy of unitary Cayley graphs and gcd-graphs, Linear Alg. Appl. 435 (2011) 1336-1343]

energy of $G_{R/(c)}$

Theorem

Let $c = p_1^{e_1} \dots p_k^{e_k}$ be factorized into a product of irreducible elements and assume that $R/(c)$ is finite.

(a) *If $e_i = 1$, then*

$$E(G_{R/(c)}(\{1, p_i\})) = 2^{k-1}|R/(p_i)||(R/(c/p_i))^*|.$$

(b) *If $e_i, e_j > 1$ (where $i \neq j$), then*

$$E(G_{R/(c)}(\{p_i, p_j\})) = 2^k|(R/(c))^*|.$$

[D. Kiani, et al, Energy of unitary Cayley graphs and gcd-graphs, Linear Alg. Appl. 435 (2011) 1336-1343]

energy of $\overline{G}_{R/(c)}$

Theorem

Assume that $c = p_1^{e_1} \dots p_k^{e_k}$ is factorized into a product of irreducible elements and $R/(c)$ is finite. Then

$$E(\overline{G}_{R/(c)}) = 2|R/(c)| - 2 + (2^k - 2)|(R/(c))^*| - \prod_{j=1}^{k} |D/(p_j)| + \prod_{j=1}^{k}(2 - |D/(p_j)|).$$

[D. Kiani, et al, Energy of unitary Cayley graphs and gcd-graphs, Linear Alg. Appl. 435 (2011) 1336-1343]

generalisations of unitary Cayley graphs

Definition

Let R be a finite commutative ring.

Let S be a subgroup of R^*, and A a subset of R^* satisfying $A^{-1} = A$.

Define $G_{R,S,A}$ to be the graph with vertex set R such that $x, y \in R$ are adjacent if and only if $x + ay \in S$ for some $a \in A$.

[K. Khashyarmanesh and M. R. Khorsandi, A generalization of the unit and unitary Cayley graphs of a commutative ring, Acta Math. Hungarica 137 (2012) 242-253]

In particular,

$$G_{R,S} := G_{R,S,\{-1\}} = \mathrm{Cay}(R, S)$$

is a 'partial' unitary Cayley graph, and

$$G_{R,S,\{1\}}$$

is a 'partial' unit graph (partial additive Cayley graph).

G_{R,R^*} is the unitary Cayley graph G_R of R.

In the same paper, properties of $G_{R,S,A}$ were discussed, especially when $S = R^*$.

However, the spectrum of $G_{R,S,A}$ is unknown in general.

integral Cayley graphs

Question

When is a Cayley graph (on an Abelian group) integral?

Definition

Let M be a nonempty finite set and $\mathcal{A}$ a family of subsets of M.

The Boolean algebra $B(\mathcal{A})$ is the lattice of subsets of M obtained by arbitrary finite unions, intersections and complements of the sets in the family $\mathcal{A}$.

Definition

In particular, for a group X, let $\mathcal{A}_X$ denote the family of subgroups of X.

Thus $B(\mathcal{A}_X)$ is the Boolean algebra generated by $\mathcal{A}_X$.

a sufficient condition

Theorem

Let X be a finite Abelian group. If $S \in B(\mathcal{A}_X)$ and $0 \notin S$, then $\mathrm{Cay}(X, S)$ *is integral.*

Conjecture

The converse is true.

[W. Klotz and T. Sander, Integral Cayley graphs over abelian groups, Elec. J. Combin. 17 (2010), R81]

The authors also gave a few families of integral Cayley graphs.

When X is cyclic, the conjecture above was proved by combining the above and the characterization of integral circulants (W. So 2005).

Theorem

A circulant graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ *is integral iff* $S \in B(\mathcal{A}_{\mathbb{Z}_n})$.

Klotz and Sander further investigated integral Cayley graphs and verified their conjecture for Abelian groups satisfying a certain condition.

[W. Klotz and T. Sander, Integral Cayley graphs defined by greatest common divisors. Elec. J. Combin. 18 (2011), P94]

cubic integral Cayley graphs

Theorem

All connected cubic integral Cayley graphs (not necessarily on Abelian groups) have been determined.

There are exactly seven such graphs up to isomorphism.

`[A. Abdollahi and E. Vatandoost, Which Cayley graphs are integral? Elec. J. Combin. 16 (2009), R122]`

The following well-known result was used in the proof of the result above.

Theorem

Let X be a finite group of order n whose irreducible characters (over $\mathbb{C}$) are $\chi_1, , \ldots, \chi_h$ with respective degrees $n_1, \ldots, n_h$.

The spectrum of the Cayley graph $\mathrm{Cay}(X, S)$ can be arranged as $\{\lambda_{ijk} : i = 1, \ldots, h; j, k = 1, \ldots, n_i\}$ such that $\lambda_{ij1} = \cdots = \lambda_{ijn_i}$ (this common value is denoted by λ_{ij}) and

$$\lambda_{i1}^t + \cdots + \lambda_{in_i}^t = \sum_{s_1, \ldots, s_t \in S} \chi_i \left(\prod_{l=1}^{t} s_l \right)$$

for any natural number t.

[L. Babai, Spectra of Cayley graphs, JCT (B) 27 (1979) 180-189]

4-regular integral Cayley graphs on Abelian groups

Theorem

Let $\mathrm{Cay}(X, S)$ *be a connected, 4-regular, integral Cayley graph on an Abelian group* X*.*

Then the order of X *belongs to a set of 24 integers between 5 and 144.*

[A. Abdollahi and E. Vatandoost, Integral quartic Cayley graphs on Abelian groups? Elec. J. Combin. 18 (2011), P89]

integral sets of a group

Definition

Let X be a finite group.

Let $\hat{X}$ be the set of characters of representations of X over $\mathbb{C}$.

A subset $A \subseteq X$ is called integral if, for every $\chi \in \hat{X}$,

$$\chi(A) := \sum_{a \in A} \chi(a)$$

is an integer.

Denote by $B(\mathcal{I}_X)$ the Boolean algebra generated by the family of integral sets of X.

`[R. C. Alperin and B. L. Peterson, Integral sets and Cayley graphs of finite groups, Elec. J. Combin. 19 (2012), P44]`

properties of integral sets

Theorem

(a) *Any atom of* $B(\mathcal{A}_X)$ *is an integral set*

(b) $B(\mathcal{A}_X) \subseteq B(\mathcal{I}_X) \subseteq \mathcal{P}(X)$, $\mathcal{P}(X)$ *being the power set of* X

(c) *Any subgroup of* X *is integral*

(d) $Y \leq X \Rightarrow B(\mathcal{I}_Y) \subseteq B(\mathcal{I}_X)$

(e) *For the dihedral group* D_n, $B(\mathcal{I}_{D_n}) = \mathcal{P}(D_n)$

Theorem

Let X *be a finite group. Then* $B(\mathcal{A}_X) = B(\mathcal{I}_X)$ *iff* X *is Abelian.*

[R. C. Alperin and B. L. Peterson, Integral sets and Cayley graphs of finite groups, Elec. J. Combin. 19 (2012), P44]

a characterization of integral Cayley graphs on Abelian groups

Theorem

Let X be a finite group. If a connected Cayley graph $\mathrm{Cay}(X, S)$ *is integral, then S is an integral set in X.*

Corollary

Let X be a finite Abelian group. A connected Cayley graph $\mathrm{Cay}(X, S)$ *is integral iff S is an integral set.*

This confirms the conjecture of Klotz and Sander which asserts that, for a finite Abelian group X and $0 \notin S = S^{-1} \subset X$, $\mathrm{Cay}(X, S)$ is integral off $S \in B(\mathcal{A}_X)$.

[R. C. Alperin and B. L. Peterson, Integral sets and Cayley graphs of finite groups, Elec. J. Combin. 19 (2012), P44]

Partial unitary Cayley graphs

Let R be a finite commutative ring and A a subgroup of R^*.

Then $\mathrm{Cay}(R, A \cup (-A))$ is a well-defined undirected Cayley graph on the additive group of R.

Problem

Investigate spectral properties of such 'partial' unitary Cayley graphs.

The special case when $R = \mathbb{Z}_n$ and $S = \{u^2 : u \in \mathbb{Z}_n^*\}$ was considered in:

[N. de Beaudrap, On restricted unitary Cayley graphs and symplectic transformations modulo n, Elec. J. Combin. 17 (2010), R69]

Examples

Example

Let $n \geq 7$ be an integer with $n \equiv 1 \mod 6$. Let a be a solution to

$$x^2 - x + 1 \equiv 0 \mod n.$$

Set $A = \langle [a] \rangle = \{\pm[1], \pm[a], \pm[a-1]\} \leq \mathbb{Z}_n^*$.

Then $A = -A$ and $\mathrm{Cay}(\mathbb{Z}_n, A)$ is a partial unitary Cayley graph.

Example

Let $n \geq 5$ be an integer with $n \equiv 1 \mod 4$. Let a be a solution to

$$x^2 + 1 \equiv 0 \mod n.$$

Set $A = \langle [a] \rangle = \{\pm[1], \pm[a]\} \leq \mathbb{Z}_n^*$.

Then $A = -A$ and $\mathrm{Cay}(\mathbb{Z}_n, A)$ is a partial unitary Cayley graph.

Cayley graphs on semidirect products

Let $K \rtimes H$ be a semidirect product of groups (e.g. a Frobenius group). Let A be an H-orbit on K.

Problem

Investigate spectral properties of $\mathrm{Cay}(K, A \cup A^{-1})$.

Example

Let $q = p^n$ be a prime power and $k \geq 2$ a divisor of $q - 1$ such that either q or $(q-1)/k$ is even. Let A be the subgroup of $\mathbb{F}_q^*$ of order $(q-1)/k$. Then $-A = A$.

The Cayley graph $\mathrm{Cay}(\mathbb{F}_q, A)$ on the additive group of $\mathbb{F}_q$ is called a generalized Paley graph (Lim and Praeger 2009).

Computing the spectra of $\mathrm{Cay}(\mathbb{F}_q, A)$ is reduced to that of Gauss sums.

Using a result of Diaconis-Stroock, we proved:

Theorem

(S. Zhou, 2009)
Let $K \rtimes H$ be a finite Frobenius group with kernel K. Let A be an H-orbit on K, where either $|H|$ is even or every element of A is an involution. Set $G = \mathrm{Cay}(K, A)$.

The spectral gap of G satisfies

$$|H| - \lambda_2(G) \geq \frac{|K|}{d \sum_{i=1}^{d} i n_i},$$

where d is the diameter of G and n_i is the number of H-orbits contained in the set of vertices at distance i from the identity of K.

thank you for your attention